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MATH TREK MATH

Unknotting knot theory

New techniques are beginning to unravel the mysteries of knots, revealing a great mathematical superstructure in the process

By Julie Rehmeyer

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Sometimes, a simple, even childish question turns out to be connected to the deepest secrets of the universe. Here's one: How many different ways can you tie your shoelaces?

Mathematicians have been puzzling over that question for a century or two, and the main thing they've discovered is that the question is really, really hard. In the last decade, though, they've developed some powerful new tools inspired by physics that have pried a few answers from the universe's clutches. Even more exciting is that the new tools seem to be the tip of a much larger theory that mathematicians are just beginning to uncover. That larger mathematical theory, if it exists, may help crack some of the hardest mathematical questions there are, questions about the mathematical structure of the three- and four-dimensional space where we live.

One of the reasons knots have given mathematicians fits is that the same knot can appear in very different guises. Tug here, tug there, and soon a knot will become unrecognizable, but remain fundamentally unchanged. To allow a knotted string to wiggle around without danger of untying, mathematicians seal its two ends together, making it a knotted circle. The first question mathematicians have to answer is simply, when are two knots really, secretly the same?

The dream is to create a sort of machine: Send in one of these looped knots, and out pops some result that would be the same regardless of the particular configuration of the knot. Because the answer wouldn't vary with the arrangement of the knot, such a machine is called a "knot invariant." And indeed, in 1927, mathematician J.W. Alexander created just such a "machine," a method that produces a polynomial (an expression like 3x2 + 4x + 1) from any knot. The good news is that Alexander's method always gives the same polynomial for a particular knot, even if the knot has been wiggled around to look very different. The bad news is that it can also give the identical answer for knots that really are different. For example, the granny knot and the square knot have identical Alexander polynomials.

Still, the Alexander polynomial was a start — and for nearly 60 years, it was about as far as mathematicians got. Then, in 1983, Vaughn Jones of the University of California, Berkeley, astonished everyone by creating a new and better knot invariant, one that could distinguish among many knots Alexander's invariant couldn't (such as the granny knot and the square knot).

The story quickly got even better. In 1988, physicist Edward Witten of the Institute for Advanced Study in Princeton turned Jones' single invariant into a whole zoo of new invariants using a link he found between Jones' method and quantum theory. The work earned Jones and Witten the Fields medal, one of the highest honors in mathematics, in 1990.

The Jones polynomial and Witten invariants were a tremendous advance, particularly because they were easy to compute. Still, they didn't unlock all the secrets of knots. "If you could make money out of telling knots apart, the Jones polynomial would be a powerful tool," says Stephen Sawin of Fairfield University. "It hasn't given a great theoretical understanding, though." In particular, just looking at the Jones polynomial of a particular knot didn't seem to reveal much about the knot or its relationship to other knots.

In the late 1990s, two sets of researchers broke through this barrier nearly simultaneously using approaches that seemed to be unrelated to one another. Mikhail Khovanov of Columbia University developed a new invariant, the Khovanov homology, using techniques from algebra. It could distinguish between any two knots the Jones polynomial could tell apart — and also some the Jones polynomial couldn't.

Meanwhile, Peter Ozsváth of Columbia University and Zoltán Szabó of Princeton University developed an invariant called knot Floer homology using techniques from symplectic geometry, a branch of geometry with close ties to physics. Their method was an improvement on the Alexander polynomial, just as Khovanov's was an improvement on the Jones polynomial. Furthermore, knot Floer homology could do something no other invariant before then had been shown to do: It could determine whether a loop was knotted at all, perfectly distinguishing an unknotted loop (called the "unknot") from any knotted one.

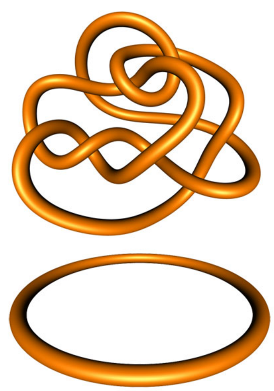
Both techniques did much more than simply distinguish knots from one another, though. Instead of producing polynomials, they produced much richer mathematical objects that helped to reveal the structure that underlies knots, the relationships among knots, and the connections between knots and other areas of mathematics. It's because of this richer structure that both techniques are called homologies. It was as if the Jones polynomial and the Alexander polynomial had been lifted to a new plane.

The techniques have begun to yield riches in information about knots. "It's turned into a bit of an industry," Ozsváth says. For example, suppose you have the power to cross the strands in a knot through one another. For a particular knot, how many times would you have to do that to unravel it? The answer is still not perfectly known, but knot Floer homology has provided much tighter bounds on that number.

Something else about the invariants has captured mathematicians' imaginations, too. Khovanov homology and knot Floer homology had very different origins, yet in the end, the two techniques seemed strikingly similar, as if they were linked at a more fundamental level than mathematicians could yet describe. "The whole study seems to be showing pieces of a single, bigger structure," Khovanov says. Khovanov homology would be one facet of that structure and knot Floer homology would be another.

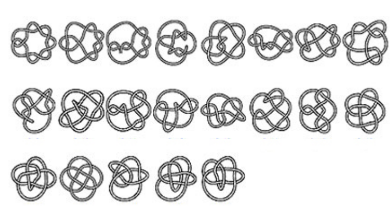
Mathematicians can already see some other facets of that great gem of a superstructure as well. All the invariants that Witten found should fit in, and perhaps even much of the mathematical architecture that underlies the Jones polynomial should too. Indeed, Sergei Gukov of the California Institute of Technology and the University of California, Santa Barbara has deepened the connection with quantum physics and string theory to “lift” the Witten invariants, just like Khovanov homology lifted the Jones polynomial. Revealing the full superstructure may be the work of a generation, Ozsváth says.

The payoff from such work may be profound. Knot Floer homology has higher-dimensional analogues that can reveal the structures of three- and four-dimensional spaces, and it is expected that Khovanov homology does as well. Four-dimensional spaces have been especially difficult to understand. Higher-dimensional spaces have enough room that complications can work themselves out, and lower-dimensional spaces are so tight that complicated behavior can’t emerge in the first place, but in four dimensions, almost anything can happen. “Understanding four dimensions would be especially exciting,” Ozsváth says, “because that’s the world we live in.”

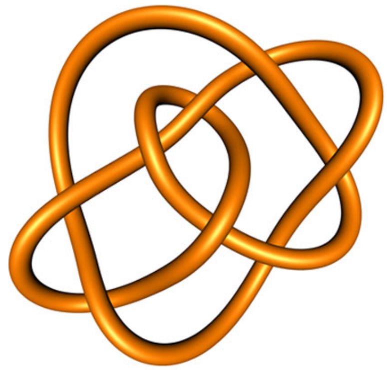


SECRETLY THE SAME Though these knots look different, mathematicians consider them the same. Twisting and pulling just the right way,without ever cutting the string, will make them look identical.

VIDEO BY ROB SCHAREIN USING SOFTWARE DOWNLOADABLE FROM WWW.KNOTPLOT.COM.



LOTS OF KNOTS This is the complete collection of knots that must cross themselves eight times. There are 49 knots with nine crossings, and 165 knots with ten crossings.



UNKNOTTING NUMBERS Until recently, no one could prove that there's no way to untangle this knot by crossing the strands through one another just once. Knot Floer homology finally provided a proof.

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